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DEVELOPMENT OF A MULTIPURPOSE LOW THRUST INTERPLANETARY TRAJECTORY CALCULATION CODE

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A multipurpose low thrust interplanetary trajectory calculation code has been developed. This code integrates the equations of motion along the trajectory assuming that the spacecraft is subject to a single attracting body and a constant thrust during both heliocentric and planetocentric phases. The histories of pitch and yaw angles for the heliocentric phase are calculated using the calculus of variations so that the spacecraft arrives at the destination planet with input heliocentric time of flight. For the planetocentric trajectory calculation, six equinoctial orbital elements are used in the vicinity of a planet with thrust direction fixed to be tangent to the path. For the heliocentric trajectory calculation, the components of position and velocity vectors are used. The output of the trajectory simulation is used as an input to mass estimating relationships that size the spacecraft. A VRML trajectory viewer helps to visualize how the spacecraft reaches its target.

Introduction

It is very likely that in the near future interplanetary missions utilizing low thrust propulsion systems will be popular because of the high specific impulse of these propulsion systems. So far there have been two spacecraft actually launched that employ ion thrusters as "main" propulsion. One is Deep Space 1 (DS-1) of the United States launched in 1998. With the thruster NSTAR (3100sec Isp, 92mN thrust), it flew by an asteroid only about 26 kilometers above the surface and observed it. DS-1 also successfully passed by comet Borrelly within 2,200 kilometers and studied this comet. Another spacecraft is MUSES-C of Japan launched in May 2003. MUSES-C carries four μ10 ion engines (>3000sec Isp, >20mN thrust with three engines). It is expected to land on the asteroid 1998SF36, execute scientific research, and then bring the samples collected at the surface of the asteroid back to the Earth in 2007.

There is no doubt that the development of a trajectory calculation code is an integral part of an interplanetary mission, since the existence of a precise code increases the possibility of the success of a mission. This is true regardless of the propulsion system.

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In this research, an interplanetary trajectory optimization code was developed for low thrust propulsion systems. The spacecraft, at first at the parking orbit around the departure planet, spirals out to escape the gravity well of the planet, and cruises around the Sun until it reaches the target planet with a desired time of flight. Then it spirals into the target parking orbit around the target planet. This code integrates the equations of motion along the trajectory, and using the calculus of variations, it calculates a optimized flight path for a constant thrust trajectory. The control variables are pitch and yaw angles, and the histories of these angles are computed. The spacecraft leaves from the parking orbit of the departure planet and arrives at the desired parking orbit around the destination planet with a user-specified time of flight.

The code is made so that the spacecraft is subject to a gravitational force from one attracting body at a time. During the simulation of planetocentric phases, the coordinate system with the planet as its origin is used, and the spacecraft is subject to the gravitational force from the planet when it is inside the sphere of influence (SOI) of the planet. Once it reaches the edge of the SOI, the heliocentric coordinate system is used and the spacecraft is assumed to be subject to the gravitational force from the Sun until it reaches the target planet. Then the coordinate system is switched to the planetocentric coordinates with the destination planet as its origin, and the spiral trajectory is calculated. The input for this code includes initial mass, specific impulse, source power of the vehicle, launch date, and heliocentric time of flight. The output includes propellant mass and ΔV. This code also has a capability of minimizing the time of flight. For a constant thrust spacecraft, minimizing the time of flight means minimizing the propellant mass, which helps decrease the vehicle's mass or increase payload mass the vehicle can deliver to the destination.

When analyzing a mission, it must be noted that the result from the trajectory calculation affects the mass property calculation, and the result from mass property calculation affects the trajectory determination, because when the vehicle's mass changes the trajectory also changes. Therefore when the mass of the spacecraft changes, a new trajectory calculation must be performed. For some missions, the initial mass at low Earth orbit (IMLEO) might be a known parameter because of the limitation of the launch vehicle's capacity. For other missions, initial mass should be calculated from a predetermined payload mass that may be determined from scientific requirements. To deal with both problems, this code can calculate either payload mass from the initial mass or initial mass from the payload mass. The maximum payload mass the spacecraft can carry is computed from the input IMLEO, or the required IMLEO is calculated so that user-specified payload mass is delivered to its destination.

It is sometimes helpful if the trajectory could be visualized. For easy visualization, the trajectory for each phase is drawn with VRML (Virtual Reality Modeling Language). The three dimensional trajectory drawing with thrust direction along the trajectory helps users understand how the spacecraft actually reaches the target. Users can pan, turn, and roll the screen, and change the viewpoint to study the trajectory from different views.

Method

The objective of this code is to calculate a constant low thrust interplanetary trajectory and to find proper settings of control variables (pitch angle and yaw angle) to reach the target planet with user-specified time of flight. This code uses the calculus of variations to calculate the trajectory by integrating the equations of motion with pitch angle and yaw angle as control variables. Users are required to input variables shown in Table 1.

Table 1: REQUIRED INPUT

Mission Specific Inputs		Vehicle Specific Inputs
Heliocentric departure date (yyyy, mm, dd)		Initial mass m_i (kg)
Heliocentric time of flight (day)		Specific impulse I_{sp} (sec)
Initial parking orbit	semi-major axis	Power level P_S (kW)
	eccentricity	System specific mass β (kg/kW)
	inclination	System efficiency η_T
Final parking orbit	semi-major axis	
	eccentricity	
	inclination	
Initial guess for the thrust vector history		

An entire trajectory is divided into three phases: the departure planetocentric phase, heliocentric phase, and arrival planetocentric phase. At first, the departure planetocentric phase is calculated. The spacecraft starts from the input initial parking orbit around the departure planet. Throughout the planetocentric phase, the thrust vector is fixed to be tangent to the path and the spacecraft spirals out until it reaches the edge of the SOI. No optimization is performed at this phase.

Next, the arrival planetocentric phase is calculated. This process is done in the same way as the departure phase such that the spacecraft spirals out until it reaches the edge of the SOI with the thrust vector tangent to the path. The actual spacecraft spirals into the target parking orbit, so this calculation simulates the spacecraft's movement backwards in time.

For a calculation of a trajectory in the vicinity of the attracting body in a planetocentric phase, a set of state variables called equinoctial orbital elements is employed rather than 6 components of position and velocity vectors. The six equinoctial elements(a, h, k, p, q, and L) are expressed by the six classical orbital elements (a, e, i, Ω, ω, and ν) as follows[11][12][13].

$$\begin{aligned}
a &= a \\
h &= e\sin(\omega+\Omega) \\
k &= e\cos(\omega+\Omega) \\
p &= \tan(i/2)\sin\Omega \\
q &= \tan(i/2)\cos\Omega \\
L &= \nu+\omega+\Omega
\end{aligned}$$

where a is the semi-major axis, e is the eccentricity, i is the inclination, Ω is the longitude of ascending node, ω is the argument of periapsis, ν is the true anomaly, and L is the true longitude. Then the time derivative of these parameters are expressed using the following equations.

$$
\begin{aligned}
\dot{a} &= \frac{2}{nG}[(ks_L - hc_L)a_{Tr} + (1 + hs_L + kc_L)a_{T\theta}] && (1)\\
\dot{h} &= \frac{G}{na(1 + hs_L + kc_L)}\{-(1 + hs_L + kc_L)c_L a_{Tr} \\
&\quad +[h + (2 + hs_L + kc_L)s_L]a_{T\theta} - k(pc_L - qs_L)a_{Th}\} && (2)\\
\dot{k} &= \frac{G}{na(1 + hs_L + kc_L)}\{(1 + hs_L + kc_L)s_L a_{Tr} \\
&\quad +[k + (2 + hs_L + kc_L)c_L]a_{T\theta} + h(pc_L - qs_L)a_{Th}\} && (3)\\
\dot{p} &= \frac{G}{2na(1 + hs_L + kc_L)}(1 + p^2 + q^2)s_L a_{Th} && (4)\\
\dot{q} &= \frac{G}{2na(1 + hs_L + kc_L)}(1 + p^2 + q^2)c_L a_{Th} && (5)\\
\dot{L} &= \frac{n(1 + hs_L + kc_L)^2}{G^3} + \frac{G}{na(1 + hs_L + kc_L)}(qs_L - pc_L)a_{Th} && (6)
\end{aligned}
$$

where μ is the planet's gravity constant, $n = (\mu/a^3)^{1/2}$ is the mean motion, and $G = (1 - h^2 - k^2)^{1/2}$. c_L and s_L represent $\cos L$ and $\sin L$, respectively. a_{Tr}, $a_{T\theta}$, and a_{Th} are the components of the acceleration vector $\vec{a}_T$ in polar coordinates.

With pitch angle α, yaw angle β, thrust level T, and spacecraft mass m, $\vec{a}_T$ is expressed as

$$
\vec{a}_T = \frac{T}{m}[\sin(\phi + \alpha)\,\cos\beta\,\hat{i}_r + \cos(\phi + \alpha)\,\cos\beta\,\hat{i}_\theta + \sin\beta\,\hat{i}_h] \qquad (7)
$$

where

$$
\begin{aligned}
\phi &= \tan^{-1}\left(\frac{e\,\sin\nu}{1 + e\cos\nu}\right) \qquad \text{is the flight-path angle}\\
e &= (h^2 + k^2)^{1/2}\\
\nu &= L - \omega - \Omega\\
\omega &= \tan^{-1}(h/k) - \tan^{-1}(p/q)\\
\Omega &= \tan^{-1}(p/q)
\end{aligned}
$$

and $\hat{i}_r$, $\hat{i}_\theta$, and $\hat{i}_h$ are the unit vectors of polar coordinates.

The merit of using this set of equations is that, when the disturbing acceleration is small, a relatively large integration step can be employed. However, because G becomes an imaginary number when the eccentricity is more than 1(because $h^2 + k^2 = e^2$), this set of equations can not be used for a trajectory with the eccentricity more than one.

When the eccentricity becomes more than one while the spacecraft is inside the SOI, the components of the position and velocity vectors in Cartesian coordinates are used as state variables. The equations of motion to be integrated are as follows.

$$f = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \\ -\mu x/r^3 + a_{Tx} \\ -\mu y/r^3 + a_{Ty} \\ -\mu z/r^3 + a_{Tz} \end{bmatrix} \tag{8}$$

where $r = (x^2+y^2+z^2)^{1/2}$, and a_{Tx}, a_{Ty}, and a_{Tz} are the components of the acceleration vector $\vec{a}_T$ from the thrust which is expressed as

$$\vec{a}_T = (T/m)(\cos\alpha\, \cos\beta\, \hat{n}_V - \sin\beta\, \hat{n}_y - \sin\alpha\, \cos\beta\, \hat{n}_z) \tag{9}$$

where

$$\begin{aligned} \hat{n}_V &= \vec{V}/|\vec{V}| \\ \hat{n}_y &= \vec{r} \times \vec{V}/|\vec{r} \times \vec{V}| \\ \hat{n}_z &= \vec{n}_V \times \vec{n}_y \end{aligned}$$

After integration of the equations of motion for the planetocentric phases, the spacecraft's velocities at the edge of the SOI's of the departure planet and the arrival planet are used as boundary conditions for the heliocentric phase. At starting point (the departure planet's position) and at the ending point (the arrival planet's position), the spacecraft's velocity with respect to the Sun, $\vec{V}^H_{s/c}$, is expressed using the spacecraft's velocity with respect to the planet, $\vec{V}^P_{s/c}$, and the planet's velocity with respect to the Sun, $\vec{V}^H_{planet}$, as follows: $\vec{V}^H_{s/c} = \vec{V}^P_{s/c} + \vec{V}^H_{planet}$.

Using $\vec{V}^H_{s/c}$ at departure and arrival as velocity constraints, and the two planets' positions as position constraints, the optimized thrust vector history is obtained using the calculus of variations. The components of the position vector (x, y, z) and the velocity vector (u, v, w) are used as state variables for the heliocentric trajectory because the eccentricity might become more than one. The equations of motion are therefore Eqs. 8 and 9, and μ is the Sun's gravitational constant.

Calculus of Variations

The computation method to obtain the histories of the control variables for a problem with functions of state variables specified at a fixed terminal time is presented below. This method is called the first-order gradient algorithm[1].

Consider the system described by the nonlinear differential equations:

$$\dot{x} = f[x(t), u(t), t], \quad x(t_0)\text{given}, \quad t_0 \le t \le t_f, \tag{10}$$

where $x(t)$, an n-vector function, is determined by $u(t)$, an m-vector function.

The optimal control problem is to find the control variables $u(t)$ on the time interval $[t_0, t_f]$ that drive the plant (10) along a trajectory $x(t)$ such that the performance index

$$J = \phi[x(t_f), t_f] \; + \; \int_{t_0}^{t_f} L[x(t), u(t), t]\, dt \tag{11}$$

$$\begin{aligned} \phi[x(t_f), t_f] \quad &: \quad \text{the final weighting function} \\ L[x(t), u(t), t] \quad &: \quad \text{the Lagrangian} \end{aligned}$$

is minimized, and such that q-vector side constraints $\psi[x(t_f), t_f]$ satisfy

$$\psi[x(t_f), t_f] = 0 \tag{12}$$

where $\phi[x(t_f), t_f]$ is the final weighting function and $L[x(t), u(t), t]$ is the Lagrangian.

To numerically solve the problem with some state variables specified at a fixed terminal time with a first-order gradient algorithm is as follows:

1. Estimate a set of control variables histories, $u(t)$.

2. Integrate the system equations $\dot{x} = f(x, u, t)$ forward with the specified initial conditions $x(t_0)$ and the control variable histories from Step 1. Record $x(t)$, $u(t)$, and $\psi[x(t_f)]$.

3. Determine an n-vector of influence functions $p(t)$, and an $(n \times q)$ matrix of influence functions, $R(t)$, by backward integration of the influence equations, using the $x(t_f)$ obtained in Step 2 to determine the boundary conditions.

   we have

$$\dot{p} = -\left(\frac{\partial f}{\partial x}\right)^T - \left(\frac{\partial L}{\partial x}\right)^T; \; p_i(t_f) = \begin{cases} 0 & i = 1, \ldots, q, \\ (\partial\phi/\partial x_i)_{t=t_f} & i = q+1, \ldots, n, \end{cases} \tag{13}$$

$$\dot{R} = -\left(\frac{\partial f}{\partial x}\right)^T; \; R_{ij}(t_f) = \begin{cases} 1, & i = j, \quad i = 1, \ldots n, \\ 0, & i \neq j, \quad j = 1, \ldots q. \end{cases} \tag{14}$$

4. Simultaneously with Step 3, compute the following integrals:

$$I_{\psi\psi} = \int_{t_i}^{t_f} R^T \frac{\partial f}{\partial u} W^{-1} \left(\frac{\partial f}{\partial u}\right)^T R \, dt \quad [(q \times q)\text{-matrix}] \tag{15}$$

$$I_{J\psi} = I_{\psi J} = \int_{t_i}^{t_f} \left(p^T \frac{\partial f}{\partial u} + \frac{\partial L}{\partial u}\right) W^{-1} \left(\frac{\partial f}{\partial u}\right)^T R \, dt \quad [q\text{-row vector}] \tag{16}$$

$$I_{JJ} = \int_{t_i}^{t_f} \left(p^T \frac{\partial f}{\partial u} + \frac{\partial L}{\partial u}\right) W^{-1} \left[\left(\frac{\partial f}{\partial u}\right)^T p + \left(\frac{\partial L}{\partial u}\right)^T\right] dt \tag{17}$$

   where W is an $(m \times \mathrm{m})$ positive-definite matrix and I_{JJ} is a scalar.

5. Choose values of $\delta\psi$ to cause the next nominal solution to be closer to the desired values $\psi[x(t_f)] = 0$. For example, one might choose $\delta\psi = -\epsilon\psi[x(t_f)], 0 < \epsilon \leq 1$. Then determine ν from $\nu = -[I_{\psi\psi}]^{-1}(\delta\psi + I_{\psi J})$.

6. Repeat Steps 1 through 6, using an improved estimate of $u(t)$, where

$$\delta u(t) = -[W(t)]^{-1} \left[\frac{\partial L}{\partial u} + [p(t) + R(t)\nu]^T \frac{\partial f}{\partial u}\right]^T. \tag{18}$$

   Stop when $\psi[x(t_f)] = 0$ and $I_{JJ} - I_{J\psi} I_{\psi\psi}^{-1} I_{\psi J} = 0$ to the desired degree of accuracy.

The method described above requires the initial guess for the control variables. Guessing the control variables is relatively easy especially if the characteristics of the problem are known. There are other methods of solving the same problem that require an initial guess for the Lagrange multipliers, but the Lagrange multipliers do not have any physical meaning and therefore are difficult to guess.

Mission Analysis

Now let us apply the above procedure to our problem. The initial conditions for the heliocentric phase are the position of the departure planet and the velocity of the spacecraft with respect to the Sun at the edge of the SOI of the departure planet. The target condition is the position of the arrival planet and the velocity of the spacecraft with respect to the Sun at the edge of the SOI of the destination planet. The goal is to minimize the difference between calculated components and target components of position and velocity vectors at the time of arrival, so the performance index, Eq. 11, can be written as

$$J = \frac{1}{2}[(x(t_f) - x_f)^2 + (y(t_f) - y_f)^2 + (z(t_f) - z_f)^2] \tag{19}$$

subject to

$$\psi(x(t_f)) = \begin{bmatrix} u(t_f) - u_f \\ v(t_f) - v_f \\ w(t_f) - w_f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

where x, y, and z are the components of the spacecraft's position and u, v, and w are the velocity components of the spacecraft, $x(t_f)$, $y(t_f)$, $z(t_f)$, $u(t_f)$, $v(t_f)$, and $w(t_f)$ are the calculated values at time t_f, and x_f, y_f, z_f, u_f, v_f, and w_f are the target values. Required matrices and partial derivatives such as $(\partial f/\partial x)$ and $(\partial L/\partial x)$ are shown in the Appendix.

When the optimization process is successfully finished, the proper output of the histories of pitch and yaw angles and the final mass of the vehicle are obtained.

Vehicle Mass Estimation

The final mass calculated from the trajectory optimization phase is used for the vehicle mass estimation. The payload mass is calculated using the calculated final mass and input values (Table 1) with the procedure shown in Table 2.

Table 2: PAYLOAD MASS ESTIMATION PROCEDURE[10]

1.	$c = g_0 \cdot I_{sp}$	exhaust velocity
2.	$P_J = \eta_T \cdot P_S$	jet power
3.	$\dot{m}_{prop} = 2P_J/c^2$	mass flow rate
4.	$T = c \cdot \dot{m}_{prop}$	thrust
5.	m_{final} from trajectory calculation	final mass
6.	$m_{prop} = m_{initial} - m_{final}$	propellant mass
7.	$m_{inert} = \beta \cdot P_S$	inert mass
8.	$m_{payload} = m_{initial} - m_{inert} - m_{prop}$	payload mass

Users can choose from the following options: (1) Calculate payload mass from input initial mass, (2) Calculate initial mass from input payload mass. For option (2), the iterative process is required. The code starts calculation with an initial guess for the initial mass and if the payload mass obtained is different from the target payload mass, the trajectory calculation as well as the procedure in Table (2) are executed iteratively. This process is performed using the bisection method by changing the initial mass estimation until the payload mass converges to the target value.

VRML Trajectory Viewer

For easy visualization, the trajectory for each phase is drawn with VRML (Virtual Reality Modeling Language). The code outputs a file that is used as a VRML input file and a three dimensional trajectory is drawn on a web browser with the thrust direction vectors shown at several points along the trajectory.

It is sometimes difficult to choose the departure date or time of flight. If a user chooses a bad combination of these two values, the calculation will not converge. Because this drawing displays the positions of departure and arrival planets, it is helpful to determine when to depart and what time of flight to choose.

Numerical Examples

Several numerical examples are presented in this section.

Payload Mass Calculation This option is used when the vehicle's initial mass is known. An example input used for the calculation is shown in Table 3.

Table 3: EXAMPLE TRAJECTORY CALCULATION INPUT

Departure Planet	Earth	Arrival Planet	Mars
Semi-major Axis a	42,238 km	Semi-major Axis a	20,000 km
Eccentricity e	0.0	Eccentricity e	0.2
Inclination i	0.0°	Inclination i	30.0°
Departure Date	5/1/2003	Heliocentric Time of Flight	200 days
Initial Mass	5,000 kg	Specific Impulse (Isp)	5,000 sec
Source Power S_J	100 kW	System Specific Mass β	0.020 kg/kW
System Efficiency η_T	0.50		

From the input in Table 3, the required values to calculate the trajectory, such as thrust, are calculated and the optimization process is executed. The results are shown in Table 4, with a propellant mass of 1117.83 kg and a payload mass of 1882.169 kg. Table 5 is the initial and target position and velocity, calculated value, and the absolute error.

Figure1 shows the trend of error for the x, y, and z components of the position during the optimization process and Figure 2 is the history of control variables. Figures 3 and 4 are the position and velocity components of the heliocentric trajectory, respectively.

As explained earlier, the VRML trajectory viewer helps to visually show how the spacecraft travels from the initial parking orbit to the final parking orbit. Figures 5,

Table 4: EXAMPLE RESULTS: PAYLOAD MASS CALCULATION

Thrust	2.03957 N	Initial Mass	5000.000 kg
Departure Spiral		Fuel Used for Heliocentric	718.820 kg
Trip Time	75.143 day	Total Propellant Mass	1117.830 kg
Fuel Consumed	270.070 kg	Final Mass	3882.169 kg
Arrival Spiral		Inert Mass	2000.000 kg
Trip Time	35.876 day	Payload Mass	1882.169 kg
Fuel Consumed	128.940 kg	Equivalent ΔV	12.407 km/s

Table 5: INITIAL CONDITION, TARGET AND CALCULATED VALUES, AND ERRORS

	Initial	Target	Calculated	Error (%)
x (AU)	-0.78784	1.34736	1.34736	0.0000
y (AU)	-0.62713	0.41923	0.41923	0.0000
z (AU)	0.00000	-0.02435	-0.02435	0.0000
u (AU/TU)	0.60805	-0.20988	-0.21023	-0.1659
v (AU/TU)	-0.78815	0.84415	0.84629	-0.2527
w (AU/TU)	0.00000	0.02283	0.02290	-0.3085

6, and 7 show the cruise trajectory during the heliocentric phase, the departure spiral trajectory escaping from the Earth's gravity, and the arrival trajectory around Mars, respectively. In Figure 5, thrust direction is shown as arrows along the trajectory. The distance between grid lines is 1AU for Figure 5, and 10DU for Figures 6 and 7. A shaded sphere for Figures 6 and 7 shows the sphere of influence.

Initial Mass Calculation Sometimes there is a case that the payload mass is a known parameter, and we want to calculate how much initial mass is required to deliver this payload to the destination planet. This example calculates the required initial mass from the input payload mass. Suppose that all the inputs except for the initial mass are the same as Table 3, and suppose the payload mass we want to carry to Low Mars Orbit is 2000.0 kg. After the iterative calculation the initial mass for this example is obtained as 5119.510 kg, as the results show in Table 6.

Minimizing the Time of Flight For a spacecraft with a constant thrust, minimizing the time of flight means minimizing fuel consumption. The next example is

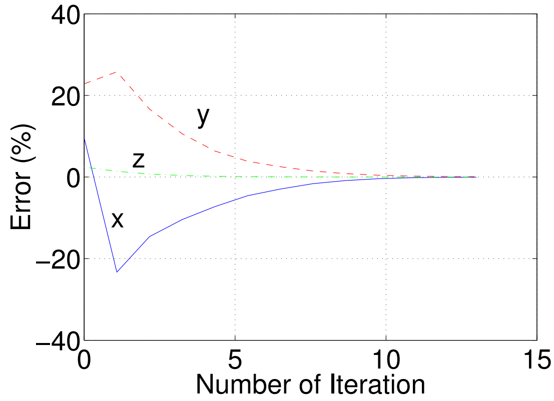

Figure 1: History of Error

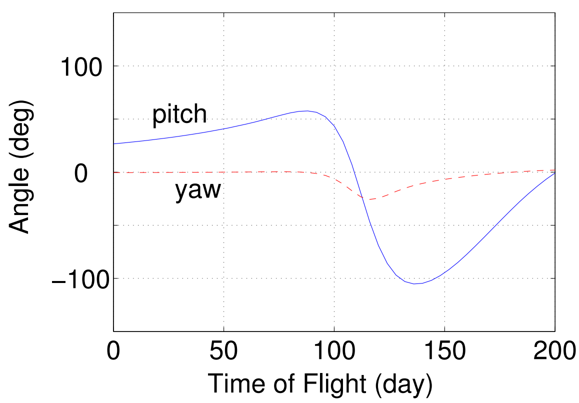

Figure 2: History of Pitch and Yaw

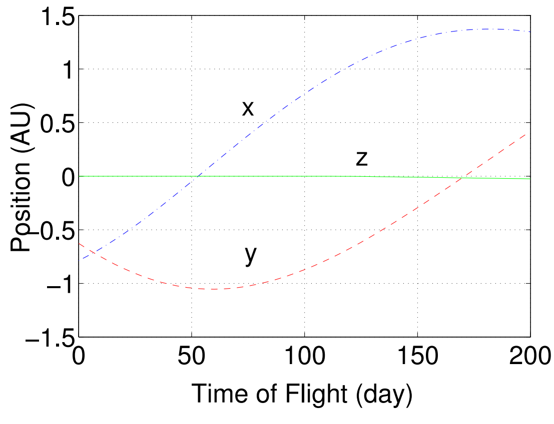

Figure 3: History of Position Components

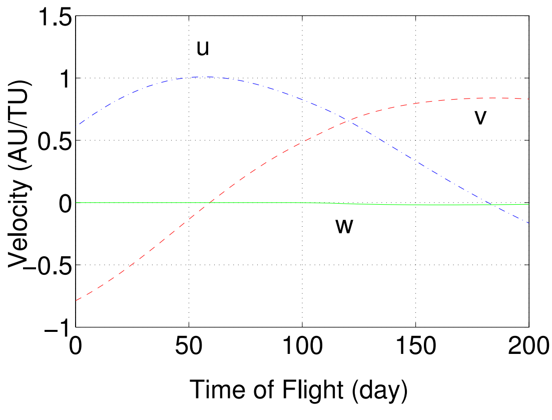

Figure 4: History of Velocity Components

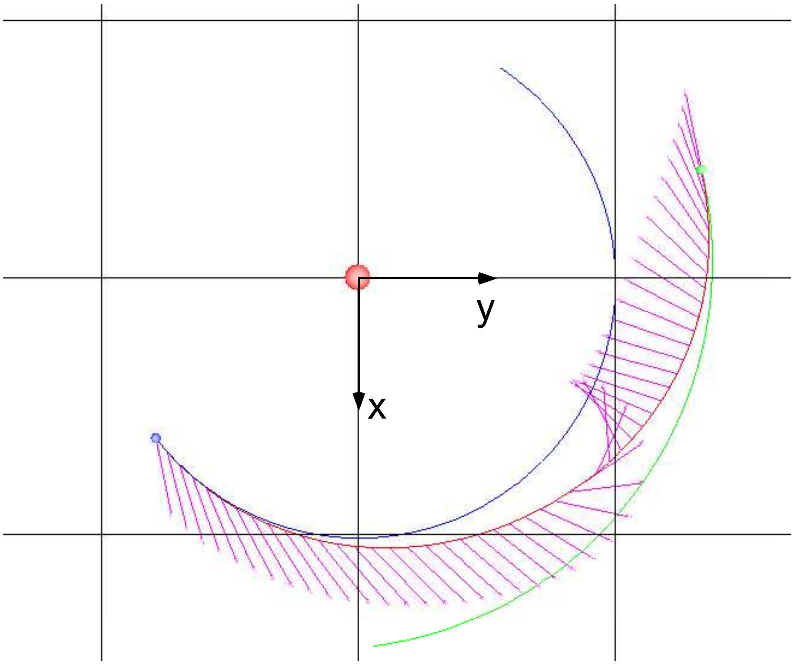

Figure 5: VRML: Heliocentric Trajectory(Earth to Mars, 200day TOF)

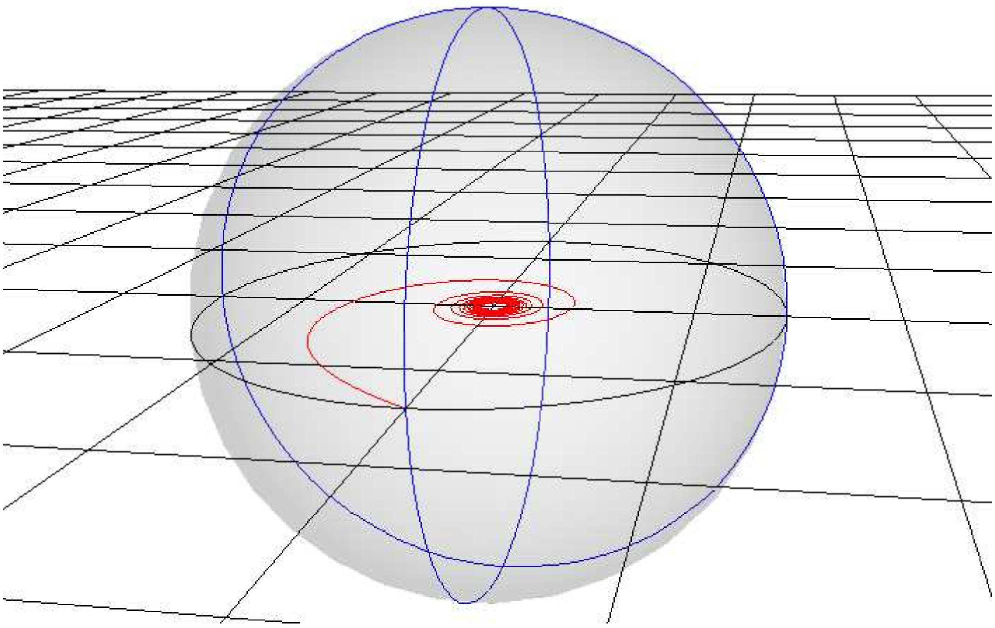
Figure 6: VRML: Geocentric Trajectory

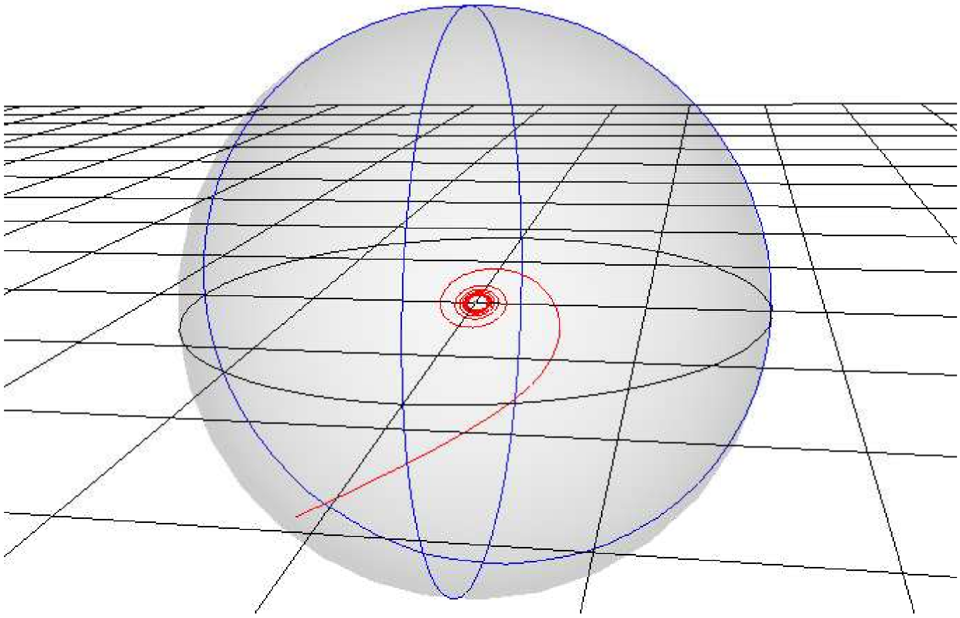

Figure 7: VRML: Trajectory at Mars Arrival

Table 6: EXAMPLE RESULTS: INITIAL MASS CALCULATION

Thrust	2.03957 N	Payload Mass	1999.993 kg
Departure Spiral		Fuel Used for Heliocentric	718.820 kg
Trip Time	76.670 day	Total Propellant Mass	1119.580 kg
Fuel Consumed	275.560 kg	Initial Mass	5119.573 kg
Arrival Spiral		Inert Mass	2000.000 kg
Trip Time	34.818 day	Final Mass	3999.993 kg
Fuel Consumed	125.200 kg	Equivalent ΔV	12.099 km/s

executed to minimize the heliocentric time of flight. Suppose all of the input data is the same as Table 3 except that the heliocentric time of flight is no longer an input. The calculated time of flight is 195.020 days, 4.98 days shorter than the results from the first example. As shown in Table 7, the required propellant mass is 1099.016 kg, 18.814 kg less than the first example.

Concluding Remarks and Future Work

A low thrust interplanetary trajectory calculation code has been developed. Using the calculus of variations, an optimal trajectory with constant thrust is obtained with pitch and yaw angles as control variables. The numerical examples showed that this code successfully optimized the heliocentric trajectory and proper settings of pitch and yaw angles are obtained. The example problems also showed that the code can be used to size a spacecraft.

Table 7: EXAMPLE RESULTS: MINIMIZING THE TIME OF FLIGHT

Thrust	2.03957 N	Heliocentric Time of Flight	195.020 day
Departure Spiral		Initial Mass	5000.000 kg
Trip Time	75.143 day	Fuel Used for Heliocentric	700.920 kg
Fuel Consumed	270.070 kg	Total Propellant Mass	1099.016 kg
Arrival Spiral		Final Mass	3900.984 kg
Trip Time	35.621 day	Inert Mass	2000.000 kg
Fuel Consumed	128.025 kg	Payload Mass	1900.984 kg
		Equivalent ΔV	12.170 km/s

In this research, the spacecraft is assumed to be subject to a single attracting body and a constant thrust for an easier calculation. For more precise results, solving the n-body problem is desirable. The program will be modified so that it can calculate the acceleration due to the gravity of more than one celestial body.

In actual missions, we might want to control the thrust level in order to achieve the best performance and to get the best trajectory. To simulate the variable thrust, the thrust level will be the third control variable rather than using a constant value throughout the trajectory as in the present results. Optimizing the trajectory by changing the thrust level will greatly improve the vehicle's performance and therefore a better solution will be obtained.

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Appendix: Matrices Used for the First-Order Gradient Algorithm

For the problem in this paper, the Lagrangian L in Eq. 11 and its derivatives are zero, so $\partial L/\partial x$ in Eq. 13 and $\partial L/\partial u$ in Eqs. 16 and 17 are zero. The matrix W is set to be an identity matrix of size $m \times m$, where m is the size of the control variable vector.

For a position vector $\vec{r} = [x\ y\ z]^T = [x_0\ x_1\ x_2]^T$ and a velocity vector $\vec{V} = [u\ v\ w]^T = [x_3\ x_4\ x_5]^T$ in the Cartesian coordinates, define the following vectors A, B, and a scaler H.

$$A = \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{bmatrix} = \begin{bmatrix} x_0(x_4^2 + x_5^2) - x_3(x_1 x_4 + x_2 x_5) \\ x_1(x_5^2 + x_3^2) - x_4(x_2 x_5 + x_0 x_3) \\ x_2(x_3^2 + x_4^2) - x_5(x_0 x_3 + x_1 x_4) \\ x_3(x_1^2 + x_2^2) - x_0(x_1 x_4 + x_2 x_5) \\ x_4(x_2^2 + x_0^2) - x_1(x_2 x_5 + x_0 x_3) \\ x_5(x_0^2 + x_1^2) - x_2(x_0 x_3 + x_1 x_4) \end{bmatrix} \tag{21}$$

$$B = \begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} x_1 x_5 - x_2 x_4 \\ x_2 x_3 - x_0 x_5 \\ x_0 x_4 - x_1 x_3 \end{bmatrix} \tag{22}$$

$$\begin{aligned} H &= |\vec{r} \times \vec{V}| \\ &= [(x_1 x_5 - x_2 x_4)^2 + (x_2 x_3 - x_0 x_5)^2 + (x_0 x_1 - x_1 x_3)^2]^{1/2} \end{aligned} \tag{23}$$

Then vectors $\hat{n}_V$, $\hat{n}_y$, and $\hat{n}_z$ are expressed as follows.

$$\hat{n}_V = \frac{\vec{V}}{|\vec{V}|} = \frac{1}{V} \begin{bmatrix} x_3 & x_4 & x_5 \end{bmatrix}^T, V = (x_3^2 + x_4^2 + x_5^2)^{1/2} \tag{24}$$

$$\hat{n}_y = \frac{\vec{r} \times \vec{V}}{|\vec{r} \times \vec{V}|} = \frac{1}{H} \begin{bmatrix} B_0 & B_1 & B_2 \end{bmatrix}^T \tag{25}$$

$$\hat{n}_z = \hat{n}_V \times \hat{n}_y = \frac{1}{VH} \begin{bmatrix} A_0 & A_1 & A_2 \end{bmatrix}^T \tag{26}$$

The thrust vector $\hat{a}_T$ is expressed as

$$\hat{a}_T = (T/m)[(\cos\alpha \cos\beta)\hat{n}_V + (-\sin\beta)\hat{n}_y + (-\sin\beta)\cos\beta)\hat{n}_z]. \tag{27}$$

where α and β are pitch angle and yaw angle, respectively, T is the thrust level, and m is the spacecraft mass.

Using the derivative of unit vectors with respect to x_i

$$\frac{\partial n_{Vi}}{\partial x_j} = \begin{bmatrix} & 1/V - x_3^2/V^3 & -x_3x_4/V^3 & -x_3x_5/V^3 \\ 0_{3\times3} & -x_3x_4/V^3 & 1/V - x_4^2/V^3 & -x_4x_5/V^3 \\ & -x_3x_5/V^3 & -x_4x_5/V^3 & 1/V - x_5^2/V^3 \end{bmatrix} \tag{28}$$

$$\frac{\partial n_{y_i}}{\partial x_j} = -\frac{1}{H}\frac{\partial B_i}{\partial x_j} + \frac{B_i}{H^2}\frac{\partial H}{\partial x_j} \tag{29}$$

$$\frac{\partial n_{z_i}}{\partial x_j} = \frac{1}{VH}\frac{\partial A_i}{\partial x_j} - \frac{A_i}{V^2H}\frac{\partial V}{\partial x_j} - \frac{A_i}{VH^2}\frac{\partial H}{\partial x_j}, \tag{30}$$

where

$$\frac{\partial B}{\partial x} = \begin{bmatrix} 0 & x_5 & -x_4 & 0 & -x_2 & x_1 \\ -x_5 & 0 & x_3 & x_2 & 0 & -x_0 \\ x_4 & -x_3 & 0 & -x_1 & x_0 & 0 \end{bmatrix}$$

$$\frac{\partial H}{\partial x} = \frac{1}{H}\begin{bmatrix} A_0 \; A_1 \; A_2 \; A_3 \; A_4 \; A_5 \end{bmatrix}^T \tag{31}$$

$$\frac{\partial A}{\partial x} = \begin{bmatrix} x_4^2 + x_5^2 & -x_3x_4 & -x_3x_5 & -x_1x_4 - x_2x_5 & 2x_0x_4 - x_1x_3 & 2x_0x_5 - x_2x_3 \\ -x_3x_4 & x_5^2 + x_3^2 & -x_4x_5 & 2x_1x_3 - x_0x_4 & -x_2x_5 - x_0x_3 & 2x_1x_5 - x_2x_4 \\ -x_3x_5 & -x_4x_5 & x_3^2 + x_4^2 & 2x_2x_3 - x_0x_5 & 2x_2x_4 - x_1x_5 & -x_0x_3 - x_1x_4 \end{bmatrix} \tag{32}$$

$$\frac{\partial V}{\partial x} = \frac{1}{V}\begin{bmatrix} 0 & 0 & 0 & x_3 & x_4 & x_5 \end{bmatrix}^T, \tag{33}$$

the derivative of a component of $\hat{a}_T$, $a_{Ti}(i = 0, 1, 2)$, with respect to $x_j(j = 0 - 5)$ is

$$\frac{\partial a_{Ti}}{\partial x_j} = (T/m)[(\cos\alpha\cos\beta)\frac{\partial n_{V_i}}{\partial x_j} + (-\sin\beta)\frac{\partial n_{y_i}}{\partial x_j} + (-\sin\beta)\frac{\partial n_{z_i}}{\partial x_j}]. \tag{34}$$

Using the derivative the an acceleration vector from the attracting body

$$\frac{\partial a_A}{\partial x} = \frac{1}{r^5}\begin{bmatrix} 3\mu x_0x_0 - \mu r^2 & 3\mu x_0x_1 & 3\mu x_0x_2 \\ 3\mu x_1x_0 & 3\mu x_1x_1 - \mu r^2 & 3\mu x_1x_2 \\ 3\mu x_2x_0 & 3\mu x_2x_1 & 3\mu x_2x_2 - \mu r^2 \end{bmatrix}, \tag{35}$$

the derivative of Eq. (8) with respect to the state variables is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} & 1 & 0 & 0 \\ 0_{3\times 3} & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ \dfrac{\partial a_{Ai}}{\partial x_j} + \dfrac{\partial a_{Ti}}{\partial x_j} & & \dfrac{\partial a_{Ti}}{\partial x_j} & \end{bmatrix}, \; i = 0, 1, 2, \; j = 0 - 5. \tag{36}$$

The derivatives of Eq. (8) with respect to the control variables $\vec{u} = [\alpha \; \beta]^T = [u_0 \; u_1]^T$ are obtained from Eq. (34).

$$\frac{\partial \vec{a}_T}{\partial u_0} = \frac{T}{m}[-\sin u_0 \cos u_1 \hat{n}_V - \cos u_0 \cos u_1 \hat{n}_z] \tag{37}$$

$$\frac{\partial \vec{a}_T}{\partial u_1} = \frac{T}{m}[-\cos u_0 \sin u_1 \hat{n}_V - \cos u_1 \hat{n}_y + \sin u_0 \sin u_1 \hat{n}_z] \tag{38}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} 0_{3\times 2} \\ \dfrac{\partial a_{Ti}}{\partial u_j} \end{bmatrix}, \; i = 0, 1, 2, \; j = 0, 1 \tag{39}$$